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Sensitive Observables of Quantum Mechanics

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Abstract

We prove that there are quantum mechanical observables which are sensitive to the type of state-vector (first type or second type) describing two correlated physical systems, in the sense that the expectation value of these 'sensitive observables' is measurably different in the two cases. The proof centers around Bell's inequality since we show that in quantum mechanics for *all* state-vectors of the second type (and only for them) sensitive observables exist in the absence of super-selection rules. Experimental verification of the existence of sensitive observables rules out local hidden variables.

1. *Introduction*

It has been proved by Bell (1965), Clauser *et al.* (1969) and Wigner that the correlation functions calculated according to *local* hidden variable theories satisfy the inequality

$$|P(a,b) - P(a,b')| \leqslant 2 - P(a',b) - P(a',b') \tag{1.1}$$

Here $P(a,b)$ is a suitably defined correlation function, related to the measurements in two separate regions of space of the dicotomic observables $A(a)$ and $B(b)$. The parameters a and b refer to the settings of the experimental apparatuses and the possible values of A and B are $A = \pm 1$, $B = \pm 1$.†

† The stronger inequality $|P(a,b) - P(a,b')| + |P(a',b) + P(a',b')| \leqslant 2$ can easily be proved. See F. Selleri, *Lettere al Nuovo Cimento*, to be published.

It is known (Bell, 1965; Clauser *et al.*, 1969; Wigner) that the quantum mechanical correlation functions, defined by

$$P(a,b) = \langle V|A(a)B(b)|V\rangle \tag{1.2}$$

do not always satisfy the inequality (1.1). In fact Bell could show that the 'singlet' state of the two observables violates (1.1).

Let a quantum mechanical system be given, consisting of two subsystems S and T. If the state-vector of the system can be written as a product

$$|V\rangle = |\Psi\rangle|\Phi\rangle \tag{1.3}$$

where $|\psi\rangle$ refers to subsystem S and $|\Phi\rangle$ to subsystem T, we say that $|V\rangle$ is a vector of the first type. If it is not possible to write $|V\rangle$ under the form (1.3) we say that it is a vector of the second type.‡

The state-vector $|V\rangle$ can always be written in terms of the state-vector $|\psi_i\rangle$ describing S and state-vector $|\Phi_i\rangle$ describing T:

$$|V\rangle = \sum_{ij} c_{ij}|\psi_i\rangle|\phi_j\rangle \tag{1.4}$$

It can be proved that the vector $|V\rangle$, given by (1.4), is of the first type if and only if the coefficients C_{ij} are factorizable, which means that one can write $C_{ij} = \alpha_i\beta_j$.

In fact, if $C_{ij} = \alpha_i\beta_j$ we can cast $|V\rangle$ under the form (1.3) where $|\psi\rangle = \Sigma_i\alpha_i|\psi_i\rangle$ and $|\Phi\rangle = \Sigma_j\beta_j|\Phi_j\rangle$.

Furthermore, if (1.3) holds, we can develop $|\psi\rangle$ over the states $|\psi_i\rangle$ with coefficients α_i and $|\Phi\rangle$ over $|\Phi_j\rangle$ with coefficients β_j and derive (1.4) with $C_{ij} = \alpha_i\beta_j$.

Now suppose we have a statistical ensemble of N identical systems S. We say that the ensemble is a pure case if all the N systems have the same state-vector.

If n_1 systems have state-vector $|\psi_1\rangle$, n_2 systems have state-vector $|\psi_2\rangle$ and so on $(n_1 + n_2 + \ldots = N)$, we say that the ensemble is a mixture of the first type. If S is a part of a larger system $\Sigma = S + T$ such that the ensemble of $N\Sigma$'s is a pure case with a state-vector of the second type (so that S as such does not have a state-vector), we say that the ensemble of systems S is a mixture of the second type. This definition can be obviously generalized to the case where the ensemble of Σ's is a mixture of the first type of state-vector of the second type.

2. *Indifferent Observables and Sensitive Observables*

In the present paragraph we will discuss ensembles of systems S which are described as mixtures of the second type, namely such that every individual system S does not have a state-vector but for which a larger

‡ This definition is strictly analogous to the one of proper and improper mixtures given by B. d'Espagnat, *Conceptions de la physique Contemporaine* (1965). Hermann, Paris.

system $\Sigma = S + T$ can be found which does. We will first restrict our discussion to ensembles of systems Σ which are pure samples.

For a given ensemble of systems $\Sigma = S + T$, such that S and T are described as belonging to a mixture of the second type, we divide the observables in two classes. We define as *indifferent* those observables whose expectation value is such that a mixture of the first type can be found giving the same expectation value. We define instead as *sensitive* those observables which are not indifferent.

The main problem of this section is, therefore, to find which observables are indifferent and which ones are sensitive. In order to solve this problem let us consider two observables A_S, A_S' of the system S and two B_T, B_T' of the system T. Obviously the operator

$$\Gamma = A_S \otimes B_T + A_S' \otimes B_T' \tag{2.1}$$

is still hermitian and can represent an observable of the system Σ. Let us prove the following theorem.

THEOREM I. *If A_S commutes with A_S' and B_T commutes with B_T' the observable Γ is indifferent.*

In fact if $|V\rangle$ is the state-vector of the system Σ we can always develop it over a set of orthonormal eigenstates $|\psi_i\rangle$ common to A_S and A_S' and, simultaneously, over a similar set of eigenstates $|\Phi_j\rangle$ common to B_T and B_T':

$$|V\rangle = \sum_{ij} c_{ij} |\psi_i\rangle |\phi_j\rangle$$

Since we are studying mixtures of the second type we must assume that the coefficients C_{ij} are not factorizable (see Section 1). This assumption is, however, not used in the proof of the theorem. Let us calculate the expectation value of Γ on the statistical ensemble.

$$\begin{aligned}\bar{\Gamma} &= \langle V|\Gamma|V\rangle \\ &= \sum_{ij}\sum_{lm} c_{ij}^* c_{lm} [\langle\psi_i|A_S|\psi_l\rangle\langle\phi_j|B_T|\phi_m\rangle + \langle\psi_i|A_S'|\psi_l\rangle\langle\phi_j|B_T'|\phi_m\rangle] \\ &= \sum_{ij}\sum_{lm} c_{ij}^* c_{lm} [a_l\,\delta_{il}\,b_m\,\delta_{jm} + a_l'\,\delta_{il}\,b_m'\,\delta_{jm}]\end{aligned} \tag{2.2}$$

where a_l, b_m, a_l', b_m' are eigenvalues of the operators A_S, B_T, A_S', B_T' respectively. From (2.2) one can easily obtain

$$\bar{\Gamma} = \sum_{ij} |c_{ij}|^2 . [a_i b_j + a_i' b_j'] \tag{2.3}$$

The same value for $\bar{\Gamma}$ could, however, be obtained with a mixture of the first type of N systems Σ, $N|C_{11}|^2$ of which with state-vector $|\psi_1\rangle|\Phi_1\rangle$, $N|C_{12}|^2$ with $|\psi_1\rangle|\Phi_2\rangle$, and so on. By definition of indifferent observable the theorem is thus proved.

From the theorem the following corollaries follow:

C1 *Any observables A_S of the system $S(B_T$ of the system $T)$ is indifferent.* In fact such an observable is a particular case of (2.1) with $B_T = I_T$, $A_S' = 0_S$, $B_T' = I_T(A_S = I_S$, $A_S' = I_S$, $B_T' = 0_T)$.

C2 *Any observable of the system Σ of the type $A_S + B_T'$ is indifferent.* This is also a particular case of (2.1) with $B_T = I_T$ and $A_S' = I_S$.

C3 *Any observable of the system Σ of the type $A_S \otimes B_T$ is indifferent.* This particular case of (2.1) can be obtained for $A_S' = 0_S$, $B_T' = I_T$.

The operators I_S, $I_T(0_S, 0_T)$ above are the unit (the zero) operators in the Hilbert spaces of the systems S and T respectively (Furry, 1936).

At this point it would look natural to try to prove that Γ, as defined in (2.1), is a sensitive observable if A_S does not commute with A_S' and/or B_T does not commute with B_T'. This is, however, not always true, as we shall see.

We will concentrate our attention on dichotomic observables and try to discover an observable difference between mixtures of the first and of the second type. Any projection operator P of Hilbert space is a dichotomic observable and has 1 and 0 as eigenvalues. We will define from such an observable a new one, D, having $+1$ and -1 as eigenvalues. Obviously

$$D = 2P - 1 \tag{2.4}$$

It is of intuitive validity and is also very simple to prove that the expectation value $\langle\eta|D|\eta\rangle$ on any state $|\eta\rangle$ or on any mixture of such states has modulus not exceeding unity.

Consider now a mixture of systems $\Sigma = S + T$ and let this be a mixture of the first type for S and T. Let then D_S, D_S' be two different dichotomic observables for S and similarly D_T, D_T' for T. We will then prove the following:

THEOREM II. *For a mixture of the first type the inequality*

$$|\overline{D_S \otimes D_T} - \overline{D_S \otimes D_T'}| + |\overline{D_S' \otimes D_T} + \overline{D_S' \otimes D_T'}| \leqslant 2 \tag{2.5}$$

(*Bell's inequality*) *is always satisfied*, if the bar denotes the expectation value.

We will start by proving first that (2.5) is satisfied on a pure sample. There it becomes

$$|\langle\eta|[D_S \otimes D_T - D_S \otimes D_T']|\eta\rangle| \\ + |\langle\eta|[D_S' \otimes D_T + D_S' \otimes D_T']|\eta\rangle| \leqslant 2 \tag{2.6}$$

If we write $|\eta\rangle = |\psi\rangle|\Phi\rangle$ where $|\psi\rangle$ describes S and $|\Phi\rangle$ describes T we get $\langle\eta|D_S \otimes D_T|\eta\rangle = \langle\psi|D_S|\psi\rangle\langle\phi|D_T|\phi\rangle$ which can be written

$$\langle\eta|D_S \otimes D_T|\eta\rangle = \bar{D}_S . \bar{D}_T$$

where

$$\bar{D}_S = \langle\psi|D_S|\psi\rangle; \qquad \bar{D}_T = \langle\phi|D_T|\phi\rangle$$

Similar expressions can be written for the other matrix elements entering in (2.6). Therefore, one can write

$$|\bar{D}_S\bar{D}_T - \bar{D}_S\bar{D}_T'| + |\bar{D}_S'\bar{D}_T + \bar{D}_S'\bar{D}_T'| \leqslant 2 \tag{2.7}$$

or

$$|\bar{D}_S|.|\bar{D}_T - \bar{D}_T'| + |\bar{D}_S'|.|\bar{D}_T + \bar{D}_T'| \leqslant 2 \tag{2.8}$$

Recalling that $|\bar{D}_S| \leqslant 1$; $|\bar{D}_S'| \leqslant 1$ we see that the inequality

$$|\bar{D}_T - \bar{D}_T'| + |\bar{D}_T + \bar{D}_T'| \leqslant 2 \tag{2.9}$$

is stronger than (2.8) in the sense that if (2.9) is satisfied then certainly so will be (2.8) (and therefore (2.5)). It is a rather simple matter to show that (2.9) is, in fact, always satisfied.

The theorem is so proved for a pure sample. Let us consider next a mixture of the first type for S and T. In such a case we have, by definition (if we have n_r systems with a state-vector $|\eta_r\rangle$ of the first type),

$$\overline{D_S\,D_T} = \sum_r \frac{n_r}{N}\langle\eta_r|D_S \otimes D_T|\eta_r\rangle \equiv \sum_r \frac{n_r}{N}\langle D_S \otimes D_T\rangle_r$$

Therefore the left-hand side of (2.5) becomes ($p_r = n_r/N$)

$$\begin{aligned}
&\left|\sum_r p_r\langle D_S \otimes D_T\rangle_r - \sum_r p_r\langle D_S \otimes D_T'\rangle_r\right| \\
&\qquad\qquad + \left|\sum_r p_r\langle D_S' \otimes D_T\rangle_r + \sum_r p_r\langle D_S' \otimes D_T'\rangle_r\right| \\
&= \left|\sum_r p_r\{\langle D_S \otimes D_T\rangle_r - \langle D_S \otimes D_T'\rangle_r\}\right| \\
&\qquad\qquad + \left|\sum_r p_r\{\langle D_S' \otimes D_T\rangle_r + \langle D_S' \otimes D_T'\rangle_r\}\right| \\
&\leqslant \sum_r p_r|\langle D_S \otimes D_T\rangle_r - \langle D_S \otimes D_T'\rangle_r| + \sum_r p_r|\langle D_S' \otimes D_T\rangle_r + \langle D_S' \otimes D_T'\rangle_r| \\
&= \sum_r p_r\{|\langle D_S \otimes D_T\rangle_r - \langle D_S \otimes D_T'\rangle_r| + |\langle D_S' \otimes D_T\rangle_r + \langle D_S' \otimes D_T'\rangle_r|\} \\
&\leqslant \sum_r p_r.2 = 2
\end{aligned}$$

As a conclusion Bell's inequality holds for an arbitrary mixture of the first type. Theorem II is fully proved.

THEOREM III. *For a mixture of the second type the observables D_S, D_T, D_S', D_T' can be chosen in such a way that the inequality*

$$|\langle D_S \otimes D_T\rangle - \langle D_S \otimes D_T'\rangle| + |\langle D_S' \otimes D_T\rangle + \langle D_S' \otimes D_T'\rangle| \leqslant 2 \tag{2.10}$$

is not satisfied if no super-selection rules are present.

We will give the proof of the theorem in a two-dimensional Hilbert space in order to simplify it. The proof will, nevertheless, be of direct physical interest in simple cases like that of spin-$\frac{1}{2}$ states, helicity states, CP-states of K^0, $\bar{K}^0$ and so on.

We have two basic vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ for system S and $|\Phi_1\rangle$ and $|\Phi_2\rangle$ for system T. If the state-vector of $S+T$ is $|\eta\rangle$ one can always write, for some choice of $|\psi_i\rangle$ and $|\Phi_j\rangle$

$$|\eta\rangle = \sum_i c_i |\psi_i\rangle\, |\phi_i\rangle \qquad (2.11)$$

In fact for some other complete set of vectors $|\bar{\Phi}_j\rangle$ one can consider the development

$$|\eta\rangle = \sum_{ij} c_{ij} |\psi_i\rangle |\phi_j\rangle$$

and put

$$\sum_j c_{ij} |\phi_j\rangle = c_i |\phi_i\rangle \qquad j = 1, 2$$

where the C_i's are some normalizing coefficients chosen in such a way that $\langle\Phi_i|\Phi_i\rangle = 1$ whence (2.11) follows.

Let us introduce the projection operators

$$\begin{aligned}
P_S &= |\psi_1\rangle\langle\psi_1| \\
P_T &= |\phi_1\rangle\langle\phi_1| \\
P_S' &= [\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle].[\alpha_1^*\langle\psi_1| + \alpha_2^*\langle\psi_2|] \\
P_T' &= [\beta_1|\phi_1\rangle + \beta_2|\phi_2\rangle].[\beta_1^*\langle\phi_1| + \beta_2^*\langle\phi_2|]
\end{aligned}$$

and the corresponding D-operators

$$D_S = 2P_S - 1; \qquad D_T = 2P_T - 1$$
$$D_S' = 2P_S' - 1; \qquad D_T' = 2P_T' - 1$$

It is a rather simple matter to show that the following results hold

$$\begin{aligned}
\langle D_S \otimes D_T\rangle &= 1 \\
\langle D_S \otimes D_T'\rangle &= |\beta_1|^2 - |\beta_2|^2 \equiv \Delta\beta \\
\langle D_S' \otimes D_T\rangle &= |\alpha_1|^2 - |\alpha_2|^2 \equiv \Delta\alpha \\
\langle D_S' \otimes D_T'\rangle &= \Delta\alpha . \Delta\beta + 8\,\mathrm{Re}\,[c_1^* c_2 \alpha_1^* \alpha_2 \beta_1^* \beta_2]
\end{aligned}$$

Notice that $-1 \leqslant \Delta\alpha \leqslant 1$ and $-1 \leqslant \Delta\beta \leqslant 1$. Therefore

$$|\langle D_S \otimes D_T\rangle - \langle D_S \otimes D_T'\rangle| = |1 - \Delta\beta| = 1 - \Delta\beta$$
$$|\langle D_S' \otimes D_T\rangle + \langle D_S' \otimes D_T'\rangle| \geqslant \Delta\alpha(1 + \Delta\beta) + 8\,\mathrm{Re}\,[c_1^* c_2 \alpha_1^* \alpha_2 \beta_1^* \beta_2]$$

Therefore, if we prove that the inequality

$$1 - \Delta\beta + \Delta\alpha(1 - \Delta\beta) + 8\,\mathrm{Re}\,[c_1^* c_2 \alpha_1^* \alpha_2 \beta_1^* \beta_2] \leqslant 2$$

can be violated, we are sure that (2.10) can be violated as well. The previous inequality can be transformed to

$$[(1 - \Delta c^2)(1 - \Delta\alpha^2)(1 - \Delta\beta^2)]^{1/2} \cos(\phi_c + \phi_\alpha + \phi_\beta) \leqslant (1 - \Delta\alpha)(1 + \Delta\beta) \qquad (2.12)$$

where ϕ_c, ϕ_α and ϕ_β are the relative phases of c_1, c_2, of α_1, α_2 and of β_1, β_2, respectively, and where $\Delta c = |c_1|^2 - |c_2|^2$ and $-1 < \Delta c < 1$ (no equalities here because we deal with a vector $|\eta\rangle$ of the second type).

One obtains easily, unless for $\Delta\alpha = 1$ and/or $\Delta\beta = -1$ (cases in which (2.12) is satisfied as an equality).

$$[1 - \Delta c^2]^{1/2}\left[\frac{1+\Delta\alpha}{1-\Delta\alpha}.\frac{1-\Delta\beta}{1+\Delta\beta}\right]^{1/2}\cos(\phi_c + \phi_\alpha + \phi_\beta) \leqslant 1 \tag{2.13}$$

Now $1 - \Delta c^2$ is a fixed numerical quantity different from zero for a given $|\eta\rangle$ of the second type. It is obvious that we can choose ϕ_α and ϕ_β in such a way that

$$\cos(\phi_c + \phi_\alpha + \phi_\beta) = 1$$

and, furthermore, that we can choose α_1, α_2, β_1, β_2 in such a way that

$$\frac{1+\Delta\alpha}{1-\Delta\alpha}.\frac{1-\Delta\beta}{1+\Delta\beta} > \frac{1}{1-\Delta c^2}$$

We conclude that (2.13) (and therefore (2.10)) can be violated for a suitable choice of the constant α_1, α_2, β_1, β_2 which amounts to a suitable choice of the observables D_S' and D_T'. The theorem is thus proved.

Notice that if $|\eta\rangle$ were of the first type one of the c_i's should necessarily vanish. This leads to $\Delta c^2 = 1$ and therefore (2.12) becomes $(1 - \Delta\alpha)(1 + \Delta\beta) \geqslant 0$ which is always satisfied, as it should be.

It follows from Theorem III that the observable

$$\Gamma_D = D_S \otimes (D_T - D_T') + D_S' \otimes (D_T + D_T') \tag{2.14}$$

is sensitive if D_S' and D_T' are chosen in a suitable way such as to violate Bell's inequality. In fact, if Γ_D as given in (2.14) were indifferent it would necessarily follow (we indicate average over a mixture of the first type with a bar)

$$\begin{aligned}
|\langle\Gamma_D\rangle| = |\bar{\Gamma}_D| &= |\overline{D_S \otimes (D_T - D_T')} + \overline{D_S' \otimes (D_T + D_T')}| \\
&\leqslant |\overline{D_S \otimes (D_T - D_T')}| + |\overline{D_S' \otimes (D_T + D_T')}| \\
&= |\overline{D_S \otimes D_T} - \overline{D_S \otimes D_T'}| + |\overline{D_S' \otimes D_T} + \overline{D_S' + D_T'}| \\
&\leqslant 2
\end{aligned}$$

the last step being justified by Theorem II. The above is like saying that our mixture of the second type satisfies Bell's inequality with the observables D_S, D_S', D_T, D_T' contrary to the previous theorem.

Therefore, the observable (2.14) is really a sensitive one. This means that one can measure $\langle\Gamma_D\rangle$ (with a suitable choice of Γ_D) over an ensemble and decide whether he is dealing with a mixture of the first or of the second type from the obtained numerical value of $\langle\Gamma_D\rangle$. This does not contradict Theorem I because D_S does not commute with D_S' (except if $\alpha_1 = 0$ or $\alpha_2 = 0$ in which cases Bell's inequality is not violated) and D_T does not commute with D_T' (again except for the cases $\beta_1 = 0$ or $\beta_2 = 0$).

That D_S and D_S' in general do not commute is easy to see. In fact

$$P_S P_S' = |\psi_1\rangle\langle\psi_1|\{\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle\}.\{\alpha_1^*\langle\psi_1| + \alpha_2^*\langle\psi_2|\}$$
$$= |\alpha_1|^2|\psi_1\rangle\langle\psi_1| + \alpha_1\alpha_2^*|\psi_1\rangle\langle\psi_2|$$

while

$$P_S' P_S = |\alpha_1|^2|\psi_1\rangle\langle\psi_1| + \alpha_1^*\alpha_2|\psi_2\rangle\langle\psi_1|$$

whence $[P_S, P_S'] = 0$ only if $\alpha_1 = 0$ or $\alpha_2 = 0$. Similarly one proves that $[P_T, P_T'] = 0$ only if $\beta_1 = 0$ or $\beta_2 = 0$.

Notice that the violation of Bell's inequality could not be obtained for all choices of α_1, α_2, β_1, β_2. In fact, if we take $\Delta\alpha < 0$ and $\Delta\beta > 0$ the inequality (2.13) is satisfied. Therefore, we conclude that there are indifferent observables of the type $\Gamma = A_S \otimes B_T + A_S' \otimes B_T'$ even if $[A_S, A_S'] \neq 0$ and/or $[B_T, B_T'] \neq 0$. Theorem I can, therefore, not be generalized to the statement 'if and *only if* $[A_S, A_S'] = 0$ and $[B_T, B_T'] = 0$ the observable Γ is indifferent'.

3. *Conclusions*

It should be stressed that our main result (Theorem III) was proved by implicitly assuming that all projection operators are observable. This is not true when super-selection rules are present and therefore mixtures of the second type may not be observably different from local hidden variable theories (in the sense of Bell's inequality) when there are super-selection rules. This turns out in fact to be the case for isotropic-spin eigenstates.

It is very interesting to notice that Jauch's definition of state of a physical system (Jauch, 1970) led him to the conclusion that mixtures of the second type do not exist. Therefore, in view of Theorem III, the experiment performed by Clauser & Freedman (1973), can be considered as the first direct experimental check of the existence of state-vectors of the second type. In view of the delicacy of this experiment and of the great conceptual value of its implications it seems very important to perform similar experiments in other cases before ruling out definitely local hidden variable theories. Second type state-vectors are responsible for many paradoxes of Quantum Mechanics (ERP, Schrödinger's cat, theory of measurement) and, in general, for the inconsistency of a realistic philosophy with the quantum mechanical axioms.†

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† For a clear discussion of this point see the quoted book by d'Espagnat.